TETRAHEDRON EQUATION AND QUANTUM R MATRICES FOR SPIN REPRESENTATIONS OF $B_n^{(1)}$, $D_n^{(1)}$ AND $D_{n+1}^{(2)}$

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Dedicated to Professor Vladimir Bazhanov on the occasion of his sixtieth birthday.

Abstract

It is known that a solution of the tetrahedron equation generates infinitely many solutions of the Yang-Baxter equation via suitable reductions. In this paper this scheme is applied to an oscillator solution of the tetrahedron equation involving bosons and fermions by using special 3d boundary conditions. The resulting solutions of the Yang-Baxter equation are identified with the quantum R matrices for the spin representations of $B_n^{(1)}, D_n^{(1)}$ and $D_{n+1}^{(2)}$.

1. Introduction

The tetrahedron equation [1, 2] is a three-dimensional (3d) extension of the Yang-Baxter (triangle) equation [3]. It is expressed as an equality between quartic products of R matrices and/or L operators, and serves as a sufficient condition for the layer to layer transfer matrices in the associated 3d lattice models to commute with each other. In general, these R matrices and L operators act on tensor product of three vector spaces reflecting the three independent directions in the 3d lattice.

Tetrahedron equations possess two notable features. First, they remain valid under n-fold composition of the L operators in one of the directions for arbitrary n. Namely they straightforwardly generalize to the n-layer situation, which is analogous to the (rather trivial) fact that a single Yang-Baxter equation implies the commutativity of row transfer matrices for arbitrary row lengths in two-dimension (2d). Second, if one of the three spaces is traced out or evaluated away appropriately, the tetrahedron equation reduces to the Yang-Baxter equation among the resulting objects. We refer to the space so masked as the "(third) hidden direction". In fact, it appears as a space of internal degrees of freedom attached to each lattice site from the resulting 2d world point of view.

Combining the above two features leads to the following fact: a solution of the tetrahedron equation generates an infinite series of solutions of the Yang-Baxter equation. This phenomenon is known as *dimension-rank* transmutation. It has been implemented earlier for a certain 3d L operator by taking the trace which corresponds to the *periodic boundary condition* in the hidden direction [4, 5, 6]. The resulting solutions of the Yang-Baxter equation have been identified with the quantum R matrices for a class of finite dimensional representations of $U_q(\widehat{sl}_n)$.

In this paper we introduce another type of boundary conditions in the hidden direction and study the resulting solutions of the Yang-Baxter equation. We start from the solution of the tetrahedron equation consisting of q-oscillator 3d R matrix and fermionic 3d L operators, which are the same as [4]. We construct special *boundary states* in a bosonic Fock space (the hidden direction) and show that they are eigenvectors of the 3d R-matrix, which is the key to make our reduction scheme to work. By evaluating the L operators with respect to these boundary states, we derive three series of solutions of the Yang-Baxter equation. Our main result is that they produce the quantum R matrices for the spin representations of $U_q(B_n^{(1)}), U_q(D_n^{(1)})$

and $U_q(D^{(2)}_{n+1})$ depending on the 3d boundary conditions. In particular we observe a curious correspondence between the Dynkin diagrams of these algebras and the boundary states in the Fock space (Remark 7.2), giving a new insight into the quantum group symmetry of the 3d integrable models.

The layout of the paper is as follows. In Section 2, we explain general schemes to obtain series of solutions to the Yang-Baxter equation from a solution of the tetrahedron equation. Section 3 presents a concrete example of 3d R matrix and 3d L operator in terms of an oscillator algebra and its Fock representation. Section 4 describes the special vectors in the Fock space which serve as 3d boundary conditions. We prove the key property that they are eigenvectors of the 3d R matrix in Proposition 4.1. In Section 5, we derive solutions $\mathcal{R}(x)$ of the Yang-Baxter equation from the 3d L operator via the reduction using the special vectors. Elements of $\mathcal{R}(x)$ are expressed in an matrix product ansatz form. Section 6 collects formulas for the spin representations of $U_q(B^{(1)}_n), U_q(D^{(1)}_n)$ [7] and $U_q(D^{(2)}_{n+1})$ and the associated quantum R matrices $R(x)$ which are necessary for the proof of our main theorem. Although $U_q(D^{(2)}_{n+1})$ case is just a slight variation of $U_q(B^{(1)}_n)$, it seems to have been treated nowhere in the literature so far. In Section 7, we present an expository proof of our main result $\mathcal{R}(x) = R(x)$ (Theorem 7.1). Depending on the choices of the special vectors (3d boundary conditions), the algebras $U_q(B^{(1)}_n), U_q(D^{(1)}_n)$ and $U_q(D^{(2)}_{n+1})$ are covered. It suggests a certain correspondence between the boundary conditions and relevant Dynkin diagrams (Remark 7.2). Our strategy of the proof is to establish that $\mathcal{R}(x)$ satisfies the standard characterization [8] of the quantum R matrix $R(x)$ [8, 9], and does not rely on explicit formulas of the matrix elements.

2. Tetrahedron equation and Yang-Baxter equation: General scheme

In this section, we explain a general scheme to generate a series of solutions to the Yang-Baxter equation from a solution of the tetrahedron equation.

Let $\mathcal{R}_{1,2,3}$ be a linear operator acting on the tensor product of three vector spaces:

$$\mathcal{R}_{1,2,3} \in \mathrm{End}(F \otimes F \otimes F) . \tag{2.1}$$

Here the space F (and V coming soon as well) can be either finite or infinite dimensional for our general discussion in this section. We call $\mathcal{R}_{1,2,3}$ (3d) R matrix. The indices in $\mathcal{R}_{1,2,3}$ are just the reminder of the three copies of F which are labeled and exhibited when preferable as

$$F \otimes F \otimes F \;=\; \overset{1}{F} \otimes \overset{2}{F} \otimes \overset{3}{F} . \tag{2.2}$$

Consider another vector space V and let $\mathcal{L}_{1,a,b}$ be an operator acting on the tensor product $F \otimes V \otimes V$, i.e.,

$$\mathcal{L}_{1,a,b} \in \mathrm{End}(F \otimes V \otimes V) , \tag{2.3}$$

where again the indices are just labels (not parameters) of the spaces as $\overset{1}{F} \otimes \overset{a}{V} \otimes \overset{b}{V}$. We call $\mathcal{L}_{1,a,b}$ (3d) L operator. A version of the quantum tetrahedron equation [4] is

$$\mathcal{R}_{1,2,3}\, \mathcal{L}_{1,a,b}\, \mathcal{L}_{2,a,c} \mathcal{L}_{3,b,c} \;=\; \mathcal{L}_{3,b,c}\, \mathcal{L}_{2,a,c}\, \mathcal{L}_{1,a,b}\, \mathcal{R}_{1,2,3} . \tag{2.4}$$

It is an equation in $\mathrm{End}(\overset{1}{F} \otimes \overset{2}{F} \otimes \overset{3}{F} \otimes \overset{a}{V} \otimes \overset{b}{V} \otimes \overset{c}{V})$. The operators act as identities on the spaces whose labels are not included in their indices. The L operators $\mathcal{L}_{1,a,b}, \mathcal{L}_{2,a,c}$ and $\mathcal{L}_{3,b,c}$ are identical except that they act nontrivially on different sets of tensor components. The relation (2.4) can be depicted as Figure 1.

We regard this as a one-layer relation. It is straightforward to generalize it to the n-layer case for any positive integer n. To do so we introduce the n-fold tensor product $\boldsymbol{V} = V^{\otimes n}$ and also attach the labels $\boldsymbol{a} = (a_1, \dots, a_n)$ for distinction to these spaces as

$$\overset{\boldsymbol{a}}{\boldsymbol{V}} = \overset{a_1}{V} \otimes \overset{a_2}{V} \otimes \cdots \otimes \overset{a_n}{V}. \tag{2.5}$$

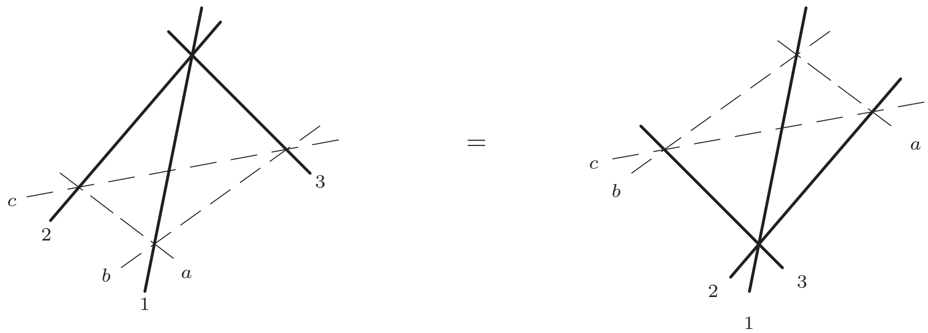


FIGURE 1. A pictorial representation of the tetrahedron equation (2.4).

Let $\overset{\boldsymbol{b}}{\boldsymbol{V}}$ and $\overset{\boldsymbol{c}}{\boldsymbol{V}}$ be copies of $\boldsymbol{V}$ with different labels $\boldsymbol{b}$ and $\boldsymbol{c}$. We compose the elementary L operators (2.3) n times as

$$\mathcal{L}_{1,\boldsymbol{a},\boldsymbol{b}} \;=\; \mathcal{L}_{1,a_1,b_1}\mathcal{L}_{1,a_2,b_2}\cdots\mathcal{L}_{1,a_n,b_n} \;\in\; \mathrm{End}(\overset{1}{F}\otimes\overset{\boldsymbol{a}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{b}}{\boldsymbol{V}}). \tag{2.6}$$

Define $\mathcal{L}_{2,\boldsymbol{a},\boldsymbol{c}}$ and $\mathcal{L}_{3,\boldsymbol{b},\boldsymbol{c}}$ similarly. They act on $\overset{1}{F}\otimes\overset{2}{F}\otimes\overset{3}{F}\otimes\overset{\boldsymbol{a}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{b}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{c}}{\boldsymbol{V}}$ nontrivially on the components specified by their indices. Then the elementary tetrahedron equation (2.4) is lifted up directly to the n-layer version:

$$\mathcal{R}_{1,2,3}\,\mathcal{L}_{1,\boldsymbol{a},\boldsymbol{b}}\,\mathcal{L}_{2,\boldsymbol{a},\boldsymbol{c}}\,\mathcal{L}_{3,\boldsymbol{b},\boldsymbol{c}} \;=\; \mathcal{L}_{3,\boldsymbol{b},\boldsymbol{c}}\,\mathcal{L}_{2,\boldsymbol{a},\boldsymbol{c}}\,\mathcal{L}_{1,\boldsymbol{a},\boldsymbol{b}}\,\mathcal{R}_{1,2,3}\,. \tag{2.7}$$

In fact, one can carry $\mathcal{R}_{1,2,3}$ through the L operators by repeated use of (2.4) from layer to layer. See Figures 2–4.

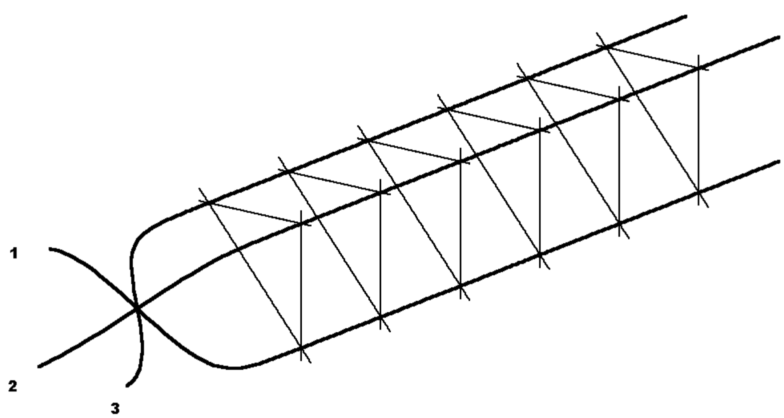


FIGURE 2. A pictorial representation of the left hand side of the tetrahedron equation (2.7).

Now we explain the prescriptions to reduce the tetrahedron equation (2.7) to the Yang-Baxter equation. The idea is to perform certain evaluations for $\mathrm{End}(F\otimes F\otimes F)$ part regarding it as "internal degrees of freedom" or a "hidden direction" perpendicular to the "vertices"

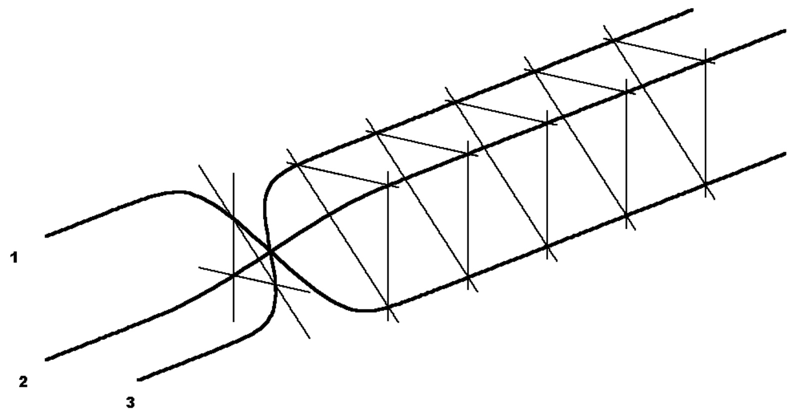


FIGURE 3. Result of application of the first elementary tetrahedron equation (2.4) to the left hand side of (2.7).

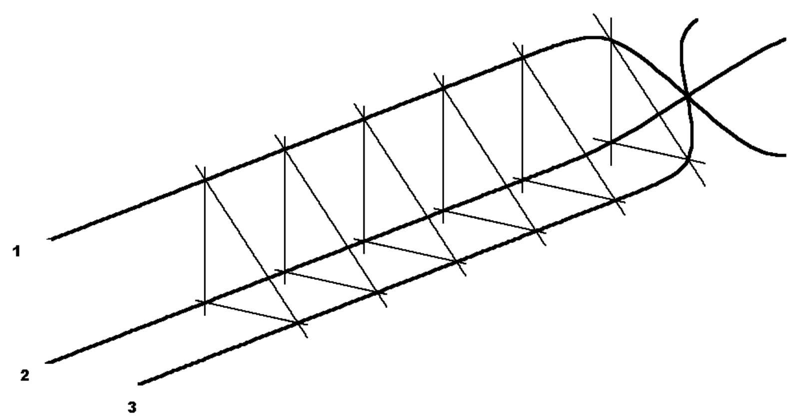


FIGURE 4. A pictorial representation of the right hand side of the tetrahedron equation (2.7).

corresponding to the other part $\mathrm{End}(\boldsymbol{V}\otimes\boldsymbol{V}\otimes\boldsymbol{V})$. The traditional way to do it is to take the trace [4]. Assuming that $\mathcal{R}_{1,2,3}$ is invertible, (2.7) leads to the Yang-Baxter equation[1]

$$\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}\,\mathcal{R}_{\boldsymbol{a},\boldsymbol{c}}\,\mathcal{R}_{\boldsymbol{b},\boldsymbol{c}} = \mathcal{R}_{\boldsymbol{b},\boldsymbol{c}}\,\mathcal{R}_{\boldsymbol{a},\boldsymbol{c}}\,\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}} \in \mathrm{End}(\overset{\boldsymbol{a}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{b}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{c}}{\boldsymbol{V}}) \tag{2.8}$$

for the R matrix defined by

$$\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}} = \mathrm{Tr}_F(\mathcal{L}_{1,\boldsymbol{a},\boldsymbol{b}}) \in \mathrm{End}(\overset{\boldsymbol{a}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{b}}{\boldsymbol{V}}), \tag{2.9}$$

where F here is actually the first copy $\overset{1}{F}$. The other ones $\mathcal{R}_{\boldsymbol{b},\boldsymbol{c}}$ and $\mathcal{R}_{\boldsymbol{a},\boldsymbol{c}}$ are defined similarly and they are identical except the nontrivially acting components in $\overset{\boldsymbol{a}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{b}}{\boldsymbol{V}}\otimes\overset{\boldsymbol{c}}{\boldsymbol{V}}$. For a concrete

[1] There is a room to include a spectral parameter by inserting a "diagonal field" in taking the trace. See [4] for detail.

result along this line, we refer to [4], which has reproduced a class quantum R matrices for $U_q(\widehat{sl}_n)$.

In this paper we consider a new scenario. Namely, suppose there are vectors

$$|\chi_s(x,y)\rangle = |\chi_s(x)\rangle \otimes |\chi_s(xy)\rangle \otimes |\chi_s(y)\rangle \in F \otimes F \otimes F\ , \tag{2.10}$$

where x, y are extra (spectral) parameters, such that

$$\mathcal{R}_{1,2,3}|\chi_s(x,y)\rangle = |\chi_s(x,y)\rangle\ . \tag{2.11}$$

The index s is a label of (possibly more than one) such vectors. Suppose also similar vectors exist in the dual space:

$$\langle\overline{\chi}_s(x,y)| = \langle\overline{\chi}_s(x)| \otimes \langle\overline{\chi}_s(xy)| \otimes \langle\overline{\chi}_s(y)| \in F^* \otimes F^* \otimes F^*\ , \tag{2.12}$$

with the property

$$\langle\overline{\chi}_s(x,y)|\mathcal{R}_{1,2,3} = \langle\overline{\chi}_s(x,y)|\ . \tag{2.13}$$

Then, evaluating the tetrahedron equation (2.7) between $\langle\overline{\chi}_s(x,y)|$ and $|\chi_t(1,1)\rangle$[2], one produces the Yang-Baxter equation

$$\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}(x)\mathcal{R}_{\boldsymbol{a},\boldsymbol{c}}(xy)\mathcal{R}_{\boldsymbol{b},\boldsymbol{c}}(y) = \mathcal{R}_{\boldsymbol{b},\boldsymbol{c}}(y)\mathcal{R}_{\boldsymbol{a},\boldsymbol{c}}(xy)\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}(x)\ \in \mathrm{End}(\overset{\boldsymbol{a}}{\boldsymbol{V}} \otimes \overset{\boldsymbol{b}}{\boldsymbol{V}} \otimes \overset{\boldsymbol{c}}{\boldsymbol{V}}), \tag{2.14}$$

which forms a series corresponding the choices $n = 1, 2, \dots$. The R matrices here are obtained from the L operator by the dual pairing of $\overset{1}{F}{}^*$ and $\overset{1}{F}$ as[3]

$$\mathcal{R}_{\boldsymbol{a},\boldsymbol{b}}(x)\,(= \mathcal{R}^{s,t}_{\boldsymbol{a},\boldsymbol{b}}(x)) = \langle\overline{\chi}_s(x)|\mathcal{L}_{1,\boldsymbol{a},\boldsymbol{b}}|\chi_t(1)\rangle \in \mathrm{End}(\overset{\boldsymbol{a}}{\boldsymbol{V}} \otimes \overset{\boldsymbol{b}}{\boldsymbol{V}})\ \text{ etc.} \tag{2.15}$$

Note the extra option regarding the choices of s and t. In fact, our main Theorem 7.1 will utilize this degree of freedom to cover the three affine Lie algebras $B^{(1)}_n, D^{(1)}_n$ and $D^{(2)}_{n+1}$ in a unified scheme. Exhibitting the dependence on s, t (and suppressing the trivial reference to the labels $\boldsymbol{a}, \boldsymbol{b}$ of the tensor components), we will write the R matrix also as $\mathcal{R}^{s,t}(x) \in \mathrm{End}(\boldsymbol{V} \otimes \boldsymbol{V})$. One may view the bra and ket vectors in (2.15) as specifying the special boundary condition along the hidden direction as in Figure 5. See also Remark 7.2.

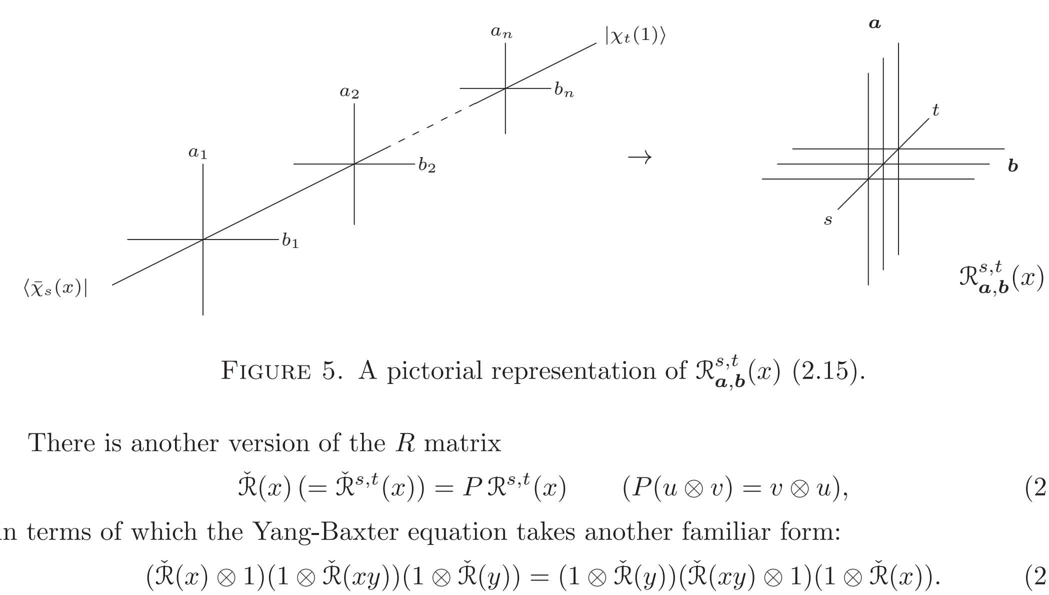


FIGURE 5. A pictorial representation of $\mathcal{R}^{s,t}_{\boldsymbol{a},\boldsymbol{b}}(x)$ (2.15).

There is another version of the R matrix

$$\check{\mathcal{R}}(x)\,(= \check{\mathcal{R}}^{s,t}(x)) = P\,\mathcal{R}^{s,t}(x) \qquad (P(u \otimes v) = v \otimes u), \tag{2.16}$$

in terms of which the Yang-Baxter equation takes another familiar form:

$$(\check{\mathcal{R}}(x) \otimes 1)(1 \otimes \check{\mathcal{R}}(xy))(1 \otimes \check{\mathcal{R}}(y)) = (1 \otimes \check{\mathcal{R}}(y))(\check{\mathcal{R}}(xy) \otimes 1)(1 \otimes \check{\mathcal{R}}(x)). \tag{2.17}$$

[2]In general, $|\chi_t(x', y')\rangle$ can be used from the right. However, in our examples treated later, such a freedom is absorbed into elsewhere and becomes equivalent to $|\chi_t(1,1)\rangle$.

[3] We regard $\mathrm{End}(\overset{\boldsymbol{a}}{\boldsymbol{V}} \otimes \overset{\boldsymbol{b}}{\boldsymbol{V}})$ here as naturally embedded into $\mathrm{End}(\overset{\boldsymbol{a}}{\boldsymbol{V}} \otimes \overset{\boldsymbol{b}}{\boldsymbol{V}} \otimes \overset{\boldsymbol{c}}{\boldsymbol{V}})$ in (2.14).

We will use the both versions $\mathcal{R}^{s,t}(x)$ and $\check{\mathcal{R}}^{s,t}(x)$ for convenience.

The rest of the paper is devoted to a concrete realization of the above scheme with the following choice:

$$\begin{aligned} &\boldsymbol{V} = V^{\otimes n}, \quad V = \mathbb{C}^2 \ \text{(fermionic Fock space)},\\ &F = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle \ \text{(bosonic Fock space)}. \end{aligned} \tag{2.18}$$

3. Oscillators and the tetrahedron equation

Let us present an example of the R matrix and L operators satisfying the tetrahedron equation (2.4). Let $\mathcal{A}$ be the associative algebra (called oscillator algebra) generated by $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ with the relations

$$\mathbf{k}\,\mathbf{a}^{\pm} = p^{\pm 1}\mathbf{a}^{\pm}\,\mathbf{k}, \quad \mathbf{a}^+\mathbf{a}^- = 1 - p^{-1}\mathbf{k}^2, \quad \mathbf{a}^-\mathbf{a}^+ = 1 - p\,\mathbf{k}^2. \tag{3.1}$$

Here p is an indeterminate. We use the representation on the bosonic Fock space $F = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle$ as follows:

$$\begin{aligned} &\mathbf{a}^+|m\rangle = \sqrt{1-p^{2m+2}}|m+1\rangle, \quad \mathbf{a}^-|m\rangle = \sqrt{1-p^{2m}}|m-1\rangle, \quad \mathbf{k}|m\rangle = p^{m+\frac{1}{2}}|m\rangle,\\ &\langle m|\mathbf{a}^- = \langle m+1|\sqrt{1-p^{2m+2}}, \quad \langle m|\mathbf{a}^+ = \langle m-1|\sqrt{1-p^{2m}}, \quad \langle m|\mathbf{k} = \langle m|p^{m+\frac{1}{2}}. \end{aligned} \tag{3.2}$$

The vector $|0\rangle$ is the total vacuum. Let $V = \mathbb{C}^2$ as in (2.18) and introduce an L operator $\mathcal{L} \in \mathrm{End}(F \otimes V \otimes V)$ by

$$\mathcal{L} = \big(\mathcal{L}(\alpha',\beta'|\alpha,\ \beta)\big)_{(\alpha',\beta'),(\alpha,\beta)}, \quad \mathcal{L}(\alpha',\beta'|\alpha,\ \beta) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -i\mathbf{k} & \mathbf{a}^+ & 0\\ 0 & \mathbf{a}^- & -i\mathbf{k} & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.3}$$

where $\mathbf{a}^{\pm}, \mathbf{k}$ are regarded as representations (3.2). The row index (α',β') and the column index (α,β) are arranged in the order $(0,0),(1,0),(0,1),(1,1)$ form top to bottom and left to right, respectively. The 0 and 1 label the base vectors of V corresponding to the fermionic states. In this interpretation, the relations (3.1) are in fact free-fermion conditions for $\mathcal{L}$ [3, 10]. The L operator (3.3) is traditionally written also as $\mathcal{L}[\mathcal{A}]$.

Let $\mathcal{A}_i$ be the copy of $\mathcal{A}$ with generators $\mathbf{a}_i^{\pm}, \mathbf{k}_i$ $(i = 1,2,3)$ that act on the ith component of $\overset{1}{F} \otimes \overset{2}{F} \otimes \overset{3}{F}$. Now one can consider the L operators

$$\mathcal{L}_{a,b}[\mathcal{A}_1],\ \mathcal{L}_{a,c}[\mathcal{A}_2],\ \mathcal{L}_{b,c}[\mathcal{A}_3] \in \mathrm{End}(\overset{1}{F} \otimes \overset{2}{F} \otimes \overset{3}{F} \otimes \overset{a}{V} \otimes \overset{b}{V} \otimes \overset{c}{V}) \tag{3.4}$$

acting nontrivially on the slots specified by their indices.

Theorem 3.1. *There is a unique (up to a constant multiple) invertible operator* $\mathcal{R} \in \mathrm{End}(F \otimes F \otimes F)$ *such that*

$$\mathcal{R}\,\mathcal{L}_{a,b}[\mathcal{A}_1]\mathcal{L}_{a,c}[\mathcal{A}_2]\mathcal{L}_{b,c}[\mathcal{A}_3] = \mathcal{L}_{b,c}[\mathcal{A}_3]\mathcal{L}_{a,c}[\mathcal{A}_2]\mathcal{L}_{a,b}[\mathcal{A}_1]\,\mathcal{R}. \tag{3.5}$$

Proof. A Proof of this theorem can be found in [4, 11]. The matrix equation (3.5) can be solved straightforwardly in the form[4]

$$\begin{cases} \mathcal{R}\ \ \mathbf{k}_2\mathbf{a}_1^{\pm}\,\mathcal{R}^{-1} \ = \ \mathbf{k}_3\mathbf{a}_1^{\pm} + \mathbf{k}_1\mathbf{a}_2^{\pm}\mathbf{a}_3^{\mp}\ ,\\ \mathcal{R}\ \ \mathbf{a}_2^{\pm}\,\mathcal{R}^{-1} \ = \ \mathbf{a}_1^{\pm}\mathbf{a}_3^{\pm} - \mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^{\pm}\ ,\\ \mathcal{R}\ \ \mathbf{k}_2\mathbf{a}_3^{\pm}\,\mathcal{R}^{-1} \ = \ \mathbf{k}_1\mathbf{a}_3^{\pm} + \mathbf{k}_3\mathbf{a}_1^{\mp}\mathbf{a}_2^{\pm}\ ,\\ \mathcal{R}\ \ \mathbf{k}_1\mathbf{k}_2\,\mathcal{R}^{-1} = \mathbf{k}_1\mathbf{k}_2,\\ \mathcal{R}\ \ \mathbf{k}_2\mathbf{k}_3\,\mathcal{R}^{-1} = \mathbf{k}_2\mathbf{k}_3. \end{cases} \tag{3.6}$$

[4]We omit a formula for $\mathcal{R}\,(\mathbf{k}_2)^2\,\mathcal{R}^{-1}$ as it will not be used in this paper. See [4].

One can easily check that (3.6) defines the automorphism of $\mathcal{A}^{\otimes 3}$. In addition, the Fock space representation is irreducible. Therefore, $\mathcal{R}$ exists and is unique up to a constant multiple. $\square$

The relation (3.5) is equivalent to quantum Korepanov equation, it can be also seen as the tetrahedral Zamolodchikov algebra/local Yang-Baxter equation for the adjoint action of $\mathcal{R}$. See the long story of [12, 13, 14, 15, 4, 11] for details. We fix the normalization of $\mathcal{R}$ by

$$\mathcal{R}|0\rangle = |0\rangle\ . \tag{3.7}$$

The relation (3.5) evidently possesses the tetrahedral structure (2.4) by the identification

$$\mathcal{L}_{a,b}[\mathcal{A}_1] = \mathcal{L}_{1,a,b}\ ,\ \mathcal{L}_{a,c}[\mathcal{A}_2] = \mathcal{L}_{2,a,c}\ ,\ \mathcal{L}_{b,c}[\mathcal{A}_3] = \mathcal{L}_{3,b,c}\ , \quad \mathcal{R} = \mathcal{R}_{1,2,3}. \tag{3.8}$$

One can verify as well, the adjoint action (3.6) of $\mathcal{R}$ coincides with the inverse adjoint action, therefore

$$\mathcal{R}^{-1} = \mathcal{R}\ . \tag{3.9}$$

An explicit formula for the matrix elements of $\mathcal{R}$ is included in Appendix A. Although, we will only need the relations (3.6) and (3.9) later in this paper.

4. Special vectors in Fock space

Let us give two vectors $|\chi_1(x,y)\rangle$ and $|\chi_2(x,y)\rangle$ in the Fock space $F\otimes F\otimes F$ (and their duals) having the properties (2.11) and (2.13), which are the key to our construction. First we introduce the following vectors in F and F^*:

$$|\chi_1(x)\rangle = \frac{1}{(x\,\mathbf{a}^+;p)_\infty}|0\rangle, \qquad |\chi_2(x)\rangle = \frac{1}{(x(\mathbf{a}^+)^2;p^4)_\infty}|0\rangle, \tag{4.1}$$

$$\langle\overline{\chi}_1(x)| = \langle 0|\frac{1}{(x\,\mathbf{a}^-;p)_\infty}, \qquad \langle\overline{\chi}_2(x)| = \langle 0|\frac{1}{(x(\mathbf{a}^-)^2;p^4)_\infty}, \tag{4.2}$$

where $(x;p)_j = \prod_{i=1}^{j}(1-xp^{i-1})$ as usual. These vectors were introduced in [16] without a proof of their properties. It is straightforward to show

$$\Big(\mathbf{a}^+ - x^{-1}(1-p^{-\frac12}\mathbf{k})\Big)|\chi_1(x)\rangle = 0, \qquad \Big(\mathbf{a}^- - x(1+p^{\frac12}\mathbf{k})\Big)|\chi_1(x)\rangle = 0, \tag{4.3}$$

$$\langle\overline{\chi}_1(x)|\Big(\mathbf{a}^- - x^{-1}(1-p^{-\frac12}\mathbf{k})\Big) = 0, \qquad \langle\overline{\chi}_1(x)|\Big(\mathbf{a}^+ - x(1+p^{\frac12}\mathbf{k})\Big) = 0, \tag{4.4}$$

$$(\mathbf{a}^- - x\mathbf{a}^+)|\chi_2(x)\rangle = 0, \qquad \langle\overline{\chi}_2(x)|(\mathbf{a}^+ - x\mathbf{a}^-) = 0. \tag{4.5}$$

By eliminating $\mathbf{k}$ in (4.3) and (4.4), we also have

$$(x^{-1}\mathbf{a}^- + px\mathbf{a}^+ - 1 - p)|\chi_1(x)\rangle = 0, \tag{4.6}$$

$$\langle\overline{\chi}_1(x)|(x^{-1}\mathbf{a}^+ + px\mathbf{a}^- - 1 - p) = 0. \tag{4.7}$$

Conversely any one of the three equations in (4.3) and (4.6) serve as characterization of $|\chi_1(x)\rangle$ up to an overall normalization. Similarly the first equation in (4.5) fixes $|\chi_2(x)\rangle$ up to an overall scalar. With regard to the dual vectors, the situation is parallel. Now the vectors $|\chi_s(x,y)\rangle$ $(s=1,2)$ and their duals are defined by

$$\begin{aligned} |\chi_s(x,y)\rangle &= |\chi_s(x)\rangle\otimes|\chi_s(xy)\rangle\otimes|\chi_s(y)\rangle \in F\otimes F\otimes F,\\ \langle\overline{\chi}_s(x,y)| &= \langle\overline{\chi}_s(x)|\otimes\langle\overline{\chi}_s(xy)|\otimes\langle\overline{\chi}_s(y)| \in F^*\otimes F^*\otimes F^*. \end{aligned} \tag{4.8}$$

Thus by setting $(u_1,u_2,u_3) = (x,xy,y)$, the vector $|\chi_1(x,y)\rangle$ is characterized by

$$(u_i\mathbf{a}_i^+ - 1 + p^{-\frac12}\mathbf{k}_i)|\chi_1(x,y)\rangle = 0 \ \text{ or} \tag{4.9}$$

$$(u_i^{-1}\mathbf{a}_i^- - 1 - p^{\frac12}\mathbf{k}_i)|\chi_1(x,y)\rangle = 0 \ \text{ or} \tag{4.10}$$

$$(u_i^{-1}\mathbf{a}_i^- + pu_i\mathbf{a}_i^+ - 1 - p)|\chi_1(x,y)\rangle = 0 \tag{4.11}$$

for each $i=1,2,3$, and so is $|\chi_2(x,y)\rangle$ by

$$(\mathbf{a}_i^- - u_i\mathbf{a}_i^+)|\chi_2(x,y)\rangle = 0 \quad (i=1,2,3) \tag{4.12}$$

in addition to the normalization $|\chi_s(x,y)\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$ + non-vacuum terms. Similar characterization holds also for $\langle\overline{\chi}_s(x,y)|$. In what follows we denote $\mathcal{R}_{1,2,3}$ simply by $\mathcal{R}$.

Proposition 4.1. *The vector $|\chi_s(x,y)\rangle$ and its dual satisfy the relations (2.11) and (2.13). Namely, the following equalities are valid for $s=1,2$:*

$$\mathcal{R}|\chi_s(x,y)\rangle = |\chi_s(x,y)\rangle, \quad \langle\overline{\chi}_s(x,y)|\mathcal{R} = \langle\overline{\chi}_s(x,y)|. \tag{4.13}$$

Proof. We shall only treat the first relation. The second one is similarly derived. It is easy to see that $\mathcal{R}|\chi_s(x,y)\rangle$ satisfies the same normalization condition as $|\chi_s(x,y)\rangle$ mentioned after (4.12).

• *Proof of* $\mathcal{R}|\chi_2(x,y)\rangle = |\chi_2(x,y)\rangle$. It then suffices to check that $\mathcal{R}|\chi_2(x,y)\rangle$ also fullfils (4.12). The $i=1$ case is shown by multiplying the invertible element $\mathcal{R}\,\mathbf{k}_2$ as

$$\begin{aligned}
&\mathcal{R}\,\mathbf{k}_2(\mathbf{a}_1^- - x\mathbf{a}_1^+)\mathcal{R}|\chi_2(x,y)\rangle \\
&\overset{(3.6)}{=} \big((\mathbf{k}_3\mathbf{a}_1^- + \mathbf{k}_3\mathbf{a}_2^-\mathbf{a}_3^+) - x(\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_3\mathbf{a}_2^+\mathbf{a}_3^-)\big)|\chi_2(x,y)\rangle \\
&= \mathbf{k}_3(\mathbf{a}_1^- - x\mathbf{a}_1^+)|\chi_2(x,y)\rangle + \mathbf{k}_3(\mathbf{a}_2^-\mathbf{a}_3^+ - x\mathbf{a}_2^+\mathbf{a}_3^-)|\chi_2(x,y)\rangle \overset{(4.12)}{=} 0.
\end{aligned}$$

The cases $i=2,3$ in (4.12) can be checked in the same manner.

• *Proof of* $\mathcal{R}|\chi_1(x,y)\rangle = |\chi_1(x,y)\rangle$. First we show $(4.11)_{i=2}$, i.e.,

$$\big((xy)^{-1}\mathbf{a}_2^- + pxy\mathbf{a}_2^+ - 1 - p\big)\mathcal{R}|\chi_1(x,y)\rangle = 0.$$

By multiplying $\mathcal{R}$ and applying (3.6), the LHS becomes

$$\big((xy)^{-1}(\mathbf{a}_1^-\mathbf{a}_3^- - \mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-) + pxy(\mathbf{a}_1^+\mathbf{a}_3^+ - \mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+) - 1 - p\big)|\chi_1(x,y)\rangle.$$

Eliminate $\mathbf{a}_i^-$ by (4.10). The result reads

$$(1+p)\big(p(1 - x\mathbf{a}_1^+)(1 - y\mathbf{a}_3^+) - \mathbf{k}_1\mathbf{k}_3\big)|\chi_1(x,y)\rangle.$$

This indeed vanishes due to (4.9). It follows that another characterizing property $(4.9)_{i=2}$, i.e., $\big(xy\mathbf{a}_2^+ - (1 - p^{-\frac{1}{2}}\mathbf{k}_2)\big)\mathcal{R}|\chi_1(x,y)\rangle = 0$ also holds. Multiplying it with $\mathcal{R}$ and using (3.6) again, we get $\big(xy(\mathbf{a}_1^+\mathbf{a}_3^+ - \mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+) - 1 + p^{-\frac{1}{2}}\mathbf{k}_2'\big)|\chi_1(x,y)\rangle = 0$, where $\mathbf{k}_2' = \mathcal{R}\,\mathbf{k}_2\mathcal{R}$. Eliminating $\mathbf{a}_1^+$ and $\mathbf{a}_3^+$ by (4.9), this is rewritten as

$$p^{-\frac{1}{2}}\big(-\mathbf{k}_1 - \mathbf{k}_3 - (p^{-\frac{1}{2}} - p^{-\frac{1}{2}})\mathbf{k}_1\mathbf{k}_3 + \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 + \mathbf{k}_2'\big)|\chi_1(x,y)\rangle = 0. \tag{4.14}$$

Now we can derive the remaining relations $(4.9)_{i=1,3}$ for $\mathcal{R}|\chi_1(x,y)\rangle$. For example in the $i=1$ case, multiplication of $\mathcal{R}\,\mathbf{k}_2$ and application of (3.6) amount to showing

$$\big(u(\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-) - \mathbf{k}_2' + p^{-\frac{1}{2}}\mathbf{k}_1\mathbf{k}_2\big)|\chi_1(x,y)\rangle = 0.$$

Rewriting $\mathbf{a}_i^\pm$ in terms of $\mathbf{k}_i$ by (4.9) and (4.10), one finds the resulting vector is proportional to the LHS of (4.14) hence zero. The equality $(4.9)_{i=3}$ can be confirmed in the same way. □

5. 2D REDUCTION OF 3D L OPERATOR

We are ready to construct R matrices satisfying the Yang-Baxter equation following the prescription (2.15). Note that the algebra $\mathcal{A}$ is naturally decomposed into the direct sum

$$\mathcal{A} = \mathcal{A}_{++} \oplus \mathcal{A}_{+-} \oplus \mathcal{A}_{-+} \oplus \mathcal{A}_{--}, \tag{5.1}$$

where $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ is the joint eigenspace of the involutive automorphism σ_1, σ_2 of $\mathcal{A}$:

$$\mathcal{A}_{\varepsilon_1,\varepsilon_2} = \{x \in \mathcal{A} \mid \sigma_i(x) = \varepsilon_i x \ (i=1,2)\}, \tag{5.2}$$

$$\sigma_1(\mathbf{a}^\pm) = -\mathbf{a}^\pm,\ \sigma_1(\mathbf{k}) = \mathbf{k}, \quad \sigma_2(\mathbf{a}^\pm) = \mathbf{a}^\pm,\ \sigma_2(\mathbf{k}) = -\mathbf{k}. \tag{5.3}$$

Using the vectors (4.1) and (4.2), we introduce the linear forms $\langle\ \rangle_{11}$, $\langle\ \rangle_{21}$ and $\langle\ \rangle_{22}$ on $\mathcal{A}$ by

$$\langle \mathcal{O}\rangle_{s1} = \frac{\langle\overline{\chi}_s(x)|\mathcal{O}|\chi_1(1)\rangle}{\langle\overline{\chi}_s(x)|\chi_1(1)\rangle} \quad (s=1,2,\ \mathcal{O}\in\mathcal{A}), \tag{5.4}$$

$$\langle \mathcal{O}\rangle_{22} = \frac{\langle\overline{\chi}_2(x)|\mathcal{O}|\chi_2(1)\rangle}{\langle\overline{\chi}_2(x)|(-i\mathbf{k})^{(1\mp1)/2}|\chi_2(1)\rangle} \quad (\mathcal{O}\in\mathcal{A}_{+\pm}\oplus\mathcal{A}_{-\pm}), \tag{5.5}$$

where $\langle\mathcal{O}\rangle_{22}=0$ for $\mathcal{O}\in\mathcal{A}_{-\pm}$. The denominators are simple factors as

$$\langle\overline{\chi}_s(x)|\chi_1(1)\rangle = \frac{(-px;p^s)_\infty}{(x;p^s)_\infty}\langle 0|0\rangle \quad (s=1,2), \tag{5.6}$$

$$\langle\overline{\chi}_2(x)|\chi_2(1)\rangle = ip^{-\frac{1}{2}}\langle\overline{\chi}_2(xp^{-2})|(-i\mathbf{k})|\chi_2(1)\rangle = \frac{(px^2;p^4)_\infty}{(x;p^4)_\infty}\langle 0|0\rangle. \tag{5.7}$$

The linear forms are evaluated explicitly by means of the standard formulas in q-analysis [17] like the q-binomial expansion and

$$\sum_{j=0}^\infty \frac{(x;p)_j}{(p;p)_j}z^j = \frac{(xz;p)_\infty}{(z;p)_\infty}.$$

The results are summarized in

Proposition 5.1. *For $j,m\in\mathbb{Z}_{\geq 0}$, the following formulas are valid:*

$$\langle(\mathbf{a}^\pm)^j\mathbf{k}^m\rangle_{11} = x^{\frac{1\pm1}{2}j}p^{\frac{m}{2}+\frac{1\mp1}{2}mj}\frac{(-p;p)_j(x;p)_m}{(-px;p)_{j+m}}. \tag{5.8}$$

$$\langle(\mathbf{a}^\pm)^j\mathbf{k}^m\rangle_{21} = p^{\frac{m}{2}\mp jm}\frac{(x;p^2)_m(p;p)_j}{(-px;p^2)_{j+m}}\sum_{i=0}^j(\pm1)^ip^{\frac{i}{2}(i-(1\pm1)j+1)}\frac{(p^{2m}x;p^2)_i(-p^{2m+2i+1}x;p^2)_{j-i}}{(p;p)_i(p;p)_{j-i}}. \tag{5.9}$$

$$\begin{aligned}\langle(\mathbf{a}^\pm)^{2j}\mathbf{k}^{2m}\rangle_{22} &= x^{\frac{1\pm1}{2}j}p^{m+(2\mp2)mj}\sum_{i=0}^j\frac{(-1)^ip^{2i^2}(p^4;p^4)_j(x;p^4)_{m+i}}{(p^4;p^4)_i(p^4;p^4)_{j-i}(xp^2;p^4)_{m+i}},\\ \langle(\mathbf{a}^\pm)^{2j}\mathbf{k}^{2m+1}\rangle_{22} &= ix^{\frac{1\pm1}{2}j}p^{m+(1\mp1)(2m+1)j}\sum_{i=0}^j\frac{(-1)^ip^{2i^2}(p^4;p^4)_j(xp^2;p^4)_{m+i}}{(p^4;p^4)_i(p^4;p^4)_{j-i}(xp^4;p^4)_{m+i}}.\end{aligned} \tag{5.10}$$

These formulas are useful for checks. However, our proof of the main Theorem 7.1 does not rely on them. We take $\boldsymbol{V}$ according to (2.18) and describe its decomposition by introducing the base vectors as

$$\boldsymbol{V} = \bigoplus_{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)\in\{0,1\}^n}\mathbb{C}v_{\boldsymbol{\alpha}}, \tag{5.11}$$

$$\boldsymbol{V} = \boldsymbol{V}_+\oplus\boldsymbol{V}_-, \qquad \boldsymbol{V}_\pm = \bigoplus_{(-1)^{\alpha_1+\cdots+\alpha_n}=\pm1}\mathbb{C}v_{\boldsymbol{\alpha}}. \tag{5.12}$$

Let x be an indeterminate (spectral parameter). The prescription (2.15) in which the n-layer L operator (2.6) is built from the basic one in (3.3) leads to the following map $\mathcal{R}^{s,t}(x):\boldsymbol{V}\otimes\boldsymbol{V}\to\boldsymbol{V}\otimes\boldsymbol{V}$ for $(s,t)=(1,1),(2,1)$ and $(2,2)$:

$$\mathcal{R}^{s,t}(x):\quad v_{\boldsymbol{\alpha}}\otimes v_{\boldsymbol{\beta}}\mapsto\sum_{\boldsymbol{\alpha}',\boldsymbol{\beta}'\in\{0,1\}^n}W_{st}\left(x\,\middle|\,\begin{matrix}\boldsymbol{\alpha}'\ \boldsymbol{\beta}'\\ \boldsymbol{\alpha}\ \boldsymbol{\beta}\end{matrix}\right)v_{\boldsymbol{\alpha}'}\otimes v_{\boldsymbol{\beta}'}, \tag{5.13}$$

$$W_{st}\left(x\,\middle|\,\begin{matrix}\boldsymbol{\alpha}'\ \boldsymbol{\beta}'\\ \boldsymbol{\alpha}\ \boldsymbol{\beta}\end{matrix}\right) = \langle\mathcal{L}(\alpha'_1,\beta'_1|\alpha_1,\beta_1)\cdots\mathcal{L}(\alpha'_n,\beta'_n|\alpha_n,\beta_n)\rangle_{st}\,, \tag{5.14}$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)$, $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_n)$, etc. See Figure 6.

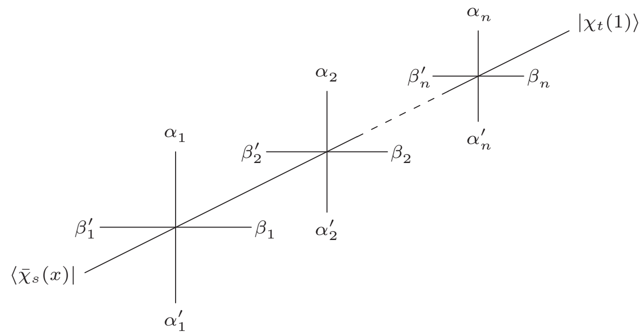


FIGURE 6. A pictorial representation of $W_{st}\left(x \,\middle|\, \begin{smallmatrix}\boldsymbol{\alpha}' & \boldsymbol{\beta}' \\ \boldsymbol{\alpha} & \boldsymbol{\beta}\end{smallmatrix}\right)$ in (5.14).

We remark that the construction (5.14) takes a matrix product ansatz form for a spin chain whose local states range over $V \otimes V$. The matrix element (5.14) depends on x through $\langle\overline{\chi}_s(x)|$ in the definitions (5.4) and (5.5). It is a rational function of $p^{\frac{1}{2}}$ and x which is normalized to be 1 for $(\boldsymbol{\alpha}', \boldsymbol{\beta}') = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ with $\boldsymbol{\alpha} = \boldsymbol{\beta}$. For $(s,t) = (2,2)$, it also equals 1 when $(\boldsymbol{\alpha}', \boldsymbol{\beta}') = (\boldsymbol{\alpha}, \boldsymbol{\beta})$ with $\boldsymbol{\alpha} - \boldsymbol{\beta} = (0, \ldots, 0, \pm 1)$ for which the corresponding product of $\mathcal{L}$'s in the bracket is $-i\mathbf{k} \in \mathcal{A}_{+-}$.

By the construction the decomposition

$$\mathcal{R}^{2,2}(x) = \mathcal{R}^{2,2}_{+,+}(x) \oplus \mathcal{R}^{2,2}_{+-}(x) \oplus \mathcal{R}^{2,2}_{-+}(x) \oplus \mathcal{R}^{2,2}_{--}(x) \tag{5.15}$$

holds, where $\mathcal{R}^{2,2}_{\varepsilon,\varepsilon'}(x) : V_\varepsilon \otimes V_{\varepsilon'} \to V_\varepsilon \otimes V_{\varepsilon'}$. As explained in Section 2, the R matrix $\mathcal{R}(x) = \mathcal{R}^{s,t}(x)$ satisfies the Yang-Baxter equation (2.14). Another version $\check{\mathcal{R}}^{s,t}(x)$ in (2.16) is described as (5.13) by replacing $v_{\boldsymbol{\alpha}'} \otimes v_{\boldsymbol{\beta}'}$ with $v_{\boldsymbol{\beta}'} \otimes v_{\boldsymbol{\alpha}'}$ in the RHS.

Example 5.2. The image of the vector $v_{0,0,0} \otimes v_{1,1,0}$ is calculated as

$$\begin{aligned}\mathcal{R}^{s,t}(x)(v_{0,0,0} \otimes v_{1,1,0}) =& \langle(\mathbf{a}^+)^2\rangle_{st} v_{1,1,0} \otimes v_{0,0,0} + \langle\mathbf{a}^+(-i\mathbf{k})\rangle_{st} v_{1,0,0} \otimes v_{0,1,0} \\ &+ \langle(-i\mathbf{k})\mathbf{a}^+\rangle_{st} v_{0,1,0} \otimes v_{1,0,0} + \langle(-i\mathbf{k})^2\rangle_{st} v_{0,0,0} \otimes v_{1,1,0}.\end{aligned}$$

The matrix elements are evaluated by using Proposition 5.1. We list the result in the table. ($\langle\mathbf{k}\,\mathbf{a}^+\rangle_{st}$ is equal to $p\langle\mathbf{a}^+\mathbf{k}\rangle_{st}$.)

	$(\mathbf{a}^+)^2$	$\mathbf{a}^+\mathbf{k}$	$\mathbf{k}^2$
$\langle\ \rangle_{21}$	$\frac{x(1+p)(1-p+px+p^3x)}{(1+px)(1+p^3x)}$	$\frac{x(1-x)p^{\frac{3}{2}}(1+p)}{(1+px)(1+p^3x)}$	$\frac{(1-x)p(1-p^2x)}{(1+px)(1+p^3x)}$
$\langle\ \rangle_{11}$	$\frac{x^2(1+p)(1+p^2)}{(1+px)(1+p^2x)}$	$\frac{x(1-x)p^{\frac{1}{2}}(1+p)}{(1+px)(1+p^2x)}$	$\frac{(1-x)p(1-px)}{(1+px)(1+p^2x)}$
$\langle\ \rangle_{22}$	$\frac{x(1-p^2)}{1-p^2x}$	0	$\frac{(1-x)p}{1-p^2x}$

6. Quantum R matrices for spin representations

Consider the quantum affine Kac-Moody algebras $U_q(B^{(1)}_n), U_q(D^{(2)}_{n+1})$ and $U_q(D^{(1)}_n)$ without the derivation operator [18, 8]. The Dynkin diagrams of $B^{(1)}_n, D^{(1)}_n$ and $D^{(2)}_{n+1}$ [19] are given in Figure 7.

Let $\{X^+_i, X^-_i, H_i | 0 \le i \le n\}$ be the Chevalley generators of $U_q(B^{(1)}_n), U_q(D^{(2)}_{n+1})$ and $U_q(D^{(1)}_n)$. In $U_q(B^{(1)}_n)$ and $U_q(D^{(2)}_{n+1})$, the classical part $\{X^+_i, X^-_i, H_i | 1 \le i \le n\}$ forms the subalgebra isomorphic to $U_q(B_n)$. We endow $\boldsymbol{V}$ (5.11) with an irreducible action of $U_q(B_n)$. For $U_q(D^{(1)}_n)$

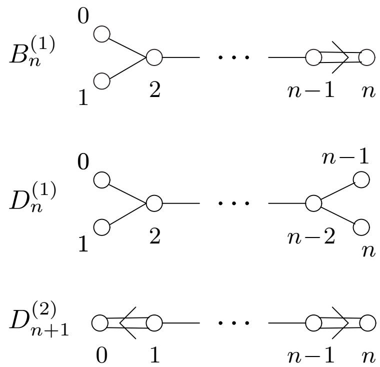


FIGURE 7. The Dynkin diagrams for $B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$ and their enumerations.

the classical part is of course $U_q(D_n)$. Each of $\boldsymbol{V}_+$ and $\boldsymbol{V}_-$ individually admits an irreducible action of $U_q(D_n)$. By further supplementing these classical parts with the "0-action" of $X_0^{\pm}$ and H_0, one gets the spin representations of $U_q(B_n^{(1)}), U_q(D_n^{(1)})$ [7] and $U_q(D_{n+1}^{(2)})$.

Let us present the concrete formulas for them. We realize $\boldsymbol{V}$ (5.11) as in (2.18). Thus we set $\boldsymbol{V} = V^{\otimes n}$ with $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ and identify the base $v_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \{0,1\}^n$ with the tensor product as

$$v_{\boldsymbol{\alpha}} = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}. \tag{6.1}$$

Introduce the 2 by 2 matrices X^+, X^- and H acting on V as

$$\begin{aligned} &X^+ v_0 = 0, \quad X^- v_0 = v_1, \quad H v_0 = \tfrac{1}{2} v_0, \\ &X^+ v_1 = v_0, \quad X^- v_1 = 0, \quad H v_1 = -\tfrac{1}{2} v_1. \end{aligned} \tag{6.2}$$

Then the action of the classical part is given as follows [7]:

$$\left.\begin{aligned} X_i^+ &= -1 \otimes \cdots \otimes 1 \otimes \overset{i}{X^+} \otimes \overset{i+1}{X^-} \otimes 1 \otimes \cdots \otimes 1 \\ H_i &= 1 \otimes \cdots \otimes 1 \otimes (\overset{i}{H} \otimes \overset{i+1}{1} - \overset{i}{1} \otimes \overset{i+1}{H}) \otimes 1 \otimes \cdots \otimes 1 \end{aligned}\right. \quad (1 \le i < n), \tag{6.3}$$

which is common to all the algebras $B_n^{(1)}, D_{n+1}^{(2)}$ and $D_n^{(1)}$. On the other hand, $i = n$ case reads

$$\begin{aligned} X_n^+ &= \frac{1}{\sqrt{q+q^{-1}}} 1 \otimes \cdots \otimes 1 \otimes X^+ \\ H_n &= 1 \otimes \cdots \otimes 1 \otimes H \end{aligned} \qquad \text{for } B_n^{(1)} \text{ and } D_{n+1}^{(2)}, \tag{6.4}$$

$$\begin{aligned} X_n^+ &= -1 \otimes \cdots \otimes 1 \otimes X^+ \otimes X^+ \\ H_n &= 1 \otimes \cdots \otimes 1 \otimes (H \otimes 1 + 1 \otimes H) \end{aligned} \qquad \text{for } D_n^{(1)}. \tag{6.5}$$

For the algebras $B_n^{(1)}$ and $D_n^{(1)}$, the 0-action is given by

$$X_0^+ = -X^- \otimes X^- \otimes 1 \otimes \cdots \otimes 1, \tag{6.6}$$

$$H_0 = -(H \otimes 1 + 1 \otimes H) \otimes 1 \otimes \cdots \otimes 1. \tag{6.7}$$

For $D_{n+1}^{(2)}$, it takes the form

$$X_0^+ = -\frac{1}{\sqrt{q+q^{-1}}} X^- \otimes 1 \otimes \cdots \otimes 1, \tag{6.8}$$

$$H_0 = -H \otimes 1 \otimes \cdots \otimes 1. \tag{6.9}$$

In any case X_i^- is given by the transpose ${}^t(X_i^+)$. We note that the base vector $v_{\boldsymbol{\alpha}}$ of $\boldsymbol{V}$ (5.11) in this paper is identified with e_μ in [7] as $v_{0,0,1} = e_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}}$ etc.

The quantum R matrix for the spin representation is a linear map $R(x) : \boldsymbol{V} \otimes \boldsymbol{V} \to \boldsymbol{V} \otimes \boldsymbol{V}$ for $B^{(1)}_n$ and $D^{(2)}_{n+1}$. Similarly, it is the linear map $R(x) : \boldsymbol{V}_\varepsilon \otimes \boldsymbol{V}_{\varepsilon'} \to \boldsymbol{V}_\varepsilon \otimes \boldsymbol{V}_{\varepsilon'}$ for $D^{(1)}_n$ for each pair of $\varepsilon, \varepsilon' \in \{+, -\}$. The composition

$$\check{R}(x) = P\,R(x) \quad (P(u \otimes v) = v \otimes u), \tag{6.10}$$

is also called R matrix. (We use the both in the sequel.) Up to an overall scalar, it is characterized by [8]

$$[\check{R}(x), \Delta(g)] = 0 \quad \text{for } g = X^{\pm}_i, H_i \ (1 \le i \le n), \tag{6.11}$$

$$\check{R}(x)(q^{H_0} \otimes X^+_0 + xX^+_0 \otimes q^{-H_0}) = (xq^{H_0} \otimes X^+_0 + X^+_0 \otimes q^{-H_0})\check{R}(x), \tag{6.12}$$

where the coproduct is specified as

$$\Delta(X^{\pm}_i) = q^{H_i} \otimes X^{\pm}_i + X^{\pm} \otimes q^{-H_i}, \quad \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i. \tag{6.13}$$

The quantum R matrix $\check{R}(x)$ satisfies the same Yang-Baxter equation (2.17) as $\check{\mathcal{R}}(x)$.

Remark 6.1. Our $\check{R}(x)$ is denoted by $R(x^{-1})$ in [7], which can be seen by comparing (6.12) here and [7, eq.(5.3)]. It is equal to $\check{R}(x^{-1})$ in the notation of [8].

As mentioned before, there are four kinds of R matrices $R(x) : \boldsymbol{V}_\varepsilon \otimes \boldsymbol{V}_{\varepsilon'} \to \boldsymbol{V}_\varepsilon \otimes \boldsymbol{V}_{\varepsilon'}$ for $D^{(1)}_n$. We gather them into a single one (2^{2n} by 2^{2n} matrix) that acts on $\boldsymbol{V} \otimes \boldsymbol{V}$ via (5.12), where the components other than $\boldsymbol{V}_\varepsilon \otimes \boldsymbol{V}_{\varepsilon'} \to \boldsymbol{V}_\varepsilon \otimes \boldsymbol{V}_{\varepsilon'}$ are to be understood as 0.

We fix the normalization of $R(x)$ by specifying a particular matrix element as

$$\begin{aligned} &R(x):\ v_{0,\dots,0} \otimes v_{0,\dots,0} \mapsto v_{0,\dots,0} \otimes v_{0,\dots,0} + \text{other terms} \quad (B^{(1)}_n, D^{(2)}_{n+1}),\\ &R(x):\ v_{0,\dots,0,\alpha} \otimes v_{0,\dots,0,\beta} \mapsto v_{0,\dots,0,\alpha} \otimes v_{0,\dots,0,\beta} + \text{other terms} \quad (D^{(1)}_n), \end{aligned} \tag{6.14}$$

where $\alpha, \beta \in \{0, 1\}$ are arbitrary. The resulting quantum R matrices will be denoted by $R_{B^{(1)}_n}(x), R_{D^{(2)}_{n+1}}(x), R_{D^{(1)}_n}(x)$. The ones obtained by them from (6.10) are similarly written as $\check{R}_{B^{(1)}_n}(x), \check{R}_{D^{(2)}_{n+1}}(x), \check{R}_{D^{(1)}_n}(x)$. They are rational functions of q and x, and admits the spectral decomposition $\check{R}(x) = \sum_{j=0}^n \rho^{(n)}_j(x) P^{(n)}_j$. To explain $P^{(n)}_j$, recall the irreducible decomposition of $U_q(B_n)$-module

$$\boldsymbol{V} \otimes \boldsymbol{V} = \boldsymbol{V}(2\Lambda_n) \oplus \boldsymbol{V}(\Lambda_{n-1}) \oplus \cdots \oplus \boldsymbol{V}(\Lambda_1) \oplus \boldsymbol{V}(0),$$

where Λ_j is the fundamental weight attached to the vertex j in the Dynkin diagram (Figure 7), and $\boldsymbol{V}(\lambda)$ denotes the irreducible $U_q(B_n)$-module with highest weight λ. (In this notation, the spin representation $\boldsymbol{V}$ on the LHS is $\boldsymbol{V}(\Lambda_n)$.) The operator $P^{(n)}_j$ in the spectral decomposition is the orthnormal projector from $\boldsymbol{V} \otimes \boldsymbol{V}$ to $\boldsymbol{V}((1+\delta_{j0})\Lambda_{n-j})$ where we set $\Lambda_0 = 0$. For $B^{(1)}_n$, $P^{(n)}_j$ is described in [7, Prop.5.1], which is actually the same also for $D^{(2)}_{n+1}$ since the two affine Lie algebras share the common classical part B_n. See Figure 7. Projectors for $D^{(1)}_n$ are also associated with similar irreducible decompositions of the $U_q(D_n)$-modules $\boldsymbol{V}_\varepsilon \otimes \boldsymbol{V}_{\varepsilon'}$. See [7] for the detail.

The eigenvalues for $B^{(1)}_n$ and $D^{(1)}_n$ are given by $\rho^{(n)}_j(x) = \tilde{\rho}^{(n)}_j(x^{-1})/\tilde{\rho}^{(n)}_{j_0}(x^{-1})$ where $\tilde{\rho}^{(n)}_j(x)$ is the one given in [7, p480, p482] with $j_0 = 0$ for $B^{(1)}_n$ and $j_0 = (1 - (-1)^j)/2$ for $D^{(1)}_n$.

The eigenvalues for $D^{(2)}_{n+1}$ seem nowhere available in the literature. Although they are not necessary for the proof of our main Theorem 7.1, we present them for reader's convenience. They are certainly vital for checks.

$$\rho^{(n)}_j(x) = \prod_{i=1}^{j} \frac{q^{2i} + (-1)^i x}{q^{2i} x + (-1)^i}. \tag{6.15}$$

We note that there are useful recursion formulas for the R matrices with respect to the rank n. The one that matches our convention for $B^{(1)}_n$ is obtained by setting $x \to x^{-1}$ in [7, Appendix] and for $D^{(1)}_n$ by setting in $q \to q^2$ in [20, sec. 2.5]. See also [21] for $q = 1$ case.

7. Main theorem

To relate $\mathcal{R}(x)$ obtained from the 3d L operators (Section 5) and $R(x)$ originating in the quantum affine algebras (Section 6), we adjust the parameter p in the oscillator algebra $\mathcal{A}$ and q in the quantum group U_q by

$$p^{\frac{1}{2}} = iq. \tag{7.1}$$

Now we state the main result of the paper.

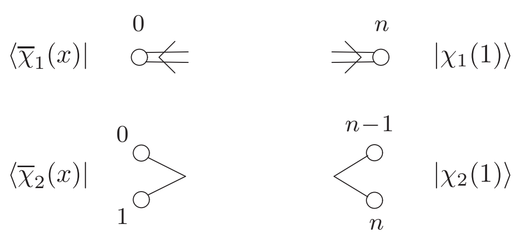
Theorem 7.1. *With the identification* (7.1)*, the following equalities are valid:*

$$\mathcal{R}^{2,1}(x) = R_{B^{(1)}_n}(x), \tag{7.2}$$

$$\mathcal{R}^{1,1}(x) = R_{D^{(2)}_{n+1}}(x), \tag{7.3}$$

$$\mathcal{R}^{2,2}(x) = R_{D^{(1)}_n}(x). \tag{7.4}$$

Remark 7.2. Comparison of these results with Figure 7 suggests the following correspondence between the boundary states $\langle\overline{\chi}_s(x)|$, $|\chi_t(1)\rangle$ in (2.15) and the end shape of the Dynkin diagrams:



In view of this, we expect that the similarly constructible $\mathcal{R}^{1,2}(x)$ yields the quantum R matrix for $U_q(B^{(1)}_n)$ corresponding to the realization of $B^{(1)}_n$ as an affinization of its another classical subalgebra D_n. We remark further that the periodic boundary condition along the F-direction in [4] corresponds to the cyclic Dynkin diagram of the relevant algebra $A^{(1)}_n$ in an analogous way to the above pictures.

Proof. In view of the normalizations, it suffices to show that $\check{\mathcal{R}}^{s,t}(x)$ (2.16) satisfies the characterization (6.11) and (6.12). For $g = H_i$ $(1 \le i \le n)$, eq. (6.11) is checked easily. Thus our first task is to show (6.11) for $g = X^+_i$. The case $g = X^-_i$ is similar and left as an exercise for the readers.

• *Proof of* (6.11) *for* $g = X^+_i$ *with* $1 \le i < n$. This case is most generic and relevant to all the algebras $B^{(1)}_n, D^{(2)}_{n+1}$ and $D^{(1)}_n$. It reads

$$\check{\mathcal{R}}^{s,t}(x)(q^{H_i} \otimes X^+_i + X^+_i \otimes q^{-H_i}) = (q^{H_i} \otimes X^+_i + X^+_i \otimes q^{-H_i})\check{\mathcal{R}}^{s,t}(x). \tag{7.5}$$

Our proof closes within the algebra $\mathcal{A}$ and is independent of its representation. It neither concerns the choice of bra and ket vectors in (5.4) and (5.5). Therefore it applies to all of $B^{(1)}_n, D^{(2)}_{n+1}$ and $D^{(1)}_n$. To illustrate the idea, we consider the matrix element of $\check{\mathcal{R}}^{s,t}(x)(q^{H_i}\otimes X^+_i)$ concerning the transition $v_{\boldsymbol{\alpha}} \otimes v_{\boldsymbol{\beta}} \mapsto v_{\boldsymbol{\beta}'} \otimes v_{\boldsymbol{\alpha}'}$. By the action of $q^{H_i} \otimes X^+_i$, the vector $v_{\boldsymbol{\alpha}} \otimes v_{\boldsymbol{\beta}}$ firstly becomes

$$(-1) \times (\cdots \otimes q^H v_{\alpha_i} \otimes q^{-H} v_{\alpha_{i+1}} \otimes \cdots) \otimes (\cdots \otimes X^+ v_{\beta_i} \otimes X^- v_{\beta_{i+1}} \otimes \cdots), \tag{7.6}$$

where the parts denoted by $\cdots$ are unchanged. See (6.3). Concretely this represents the following vector in $\boldsymbol{V}\otimes\boldsymbol{V}$:

$$-\delta_{\beta_i 1}\delta_{\beta_{i+1}0}q^{(\frac{1}{2}-\alpha_i)-(\frac{1}{2}-\alpha_{i+1})}(\cdots\otimes v_{\alpha_i}\otimes v_{\alpha_{i+1}}\otimes\cdots)\otimes(\cdots\otimes v_0\otimes v_1\otimes\cdots). \tag{7.7}$$

See (6.2). After further applying $\check{\mathcal{R}}^{s,t}(x)$ to this, the coefficient of $v_{\boldsymbol{\beta}'}\otimes v_{\boldsymbol{\alpha}'}$ in the resulting vector is 0 unless $(\alpha'_j,\beta'_j)=(\alpha_j,\beta_j)$ for all $j\neq i,i+1$. If this condition is met, the matrix element under consideration takes the form

$$\langle ML(\alpha'_i,\beta'_i|q^H\alpha_i,X^+\beta_i)L(\alpha'_{i+1},\beta'_{i+1}|q^{-H}\alpha_{i+1},X^-\beta_{i+1})N\rangle_{st} \tag{7.8}$$

for some elements $M,N\in\mathcal{A}$. Here and in what follows, we employ the slightly abused notation like

$$\begin{aligned}L(\alpha'_i,\beta'_i|q^H\alpha_i,X^+\beta_i)&=q^{\frac{1}{2}-\alpha_i}\delta_{\beta_i 1}\mathcal{L}(\alpha'_i,\beta'_i|\alpha_i,0),\\ L(q^{-H}\alpha'_i,X^-\beta'_i|\alpha_i,\beta_i)&=q^{-\frac{1}{2}+\alpha'_i}\delta_{\beta'_i 0}\mathcal{L}(\alpha'_i,1|\alpha_i,\beta_i),\end{aligned} \tag{7.9}$$

where $\mathcal{L}(\alpha',\beta'|\alpha,\beta)$ is given by (3.3). (Recall that the action of $\check{\mathcal{R}}^{s,t}(x)$ is described by (5.13) and (5.14) followed by the transposition P as in (2.16).) By similar calculations, one finds that the matrix element of LHS-RHS of (7.5) concerning $v_{\boldsymbol{\alpha}}\otimes v_{\boldsymbol{\beta}}\mapsto v_{\boldsymbol{\beta}'}\otimes v_{\boldsymbol{\alpha}'}$ is proportional to $\langle MZ_iN\rangle_{st}$ with

$$\begin{aligned}Z_i=&L(\alpha'_i,\beta'_i|q^H\alpha_i,X^+\beta_i)L(\alpha'_{i+1},\beta'_{i+1}|q^{-H}\alpha_{i+1},X^-\beta_{i+1})\\ &+L(\alpha'_i,\beta'_i|X^+\alpha_i,q^{-H}\beta_i)L(\alpha'_{i+1},\beta'_{i+1}|X^-\alpha_{i+1},q^H\beta_{i+1})\\ &-L(X^-\alpha'_i,q^H\beta'_i|\alpha_i,\beta_i)L(X^+\alpha'_{i+1},q^{-H}\beta'_{i+1}|\alpha_{i+1},\beta_{i+1})\\ &-L(q^{-H}\alpha'_i,X^-\beta'_i|\alpha_i,\beta_i)L(q^H\alpha'_{i+1},X^+\beta'_{i+1}|\alpha_{i+1},\beta_{i+1})\in\mathcal{A}.\end{aligned} \tag{7.10}$$

The four terms here correspond to those in (7.5). Note for example in the composition $(q^{H_i}\otimes X_i^+)\check{\mathcal{R}}^{s,t}(x)$, one needs to look at the transition $\check{\mathcal{R}}^{s,t}(x): v_{\boldsymbol{\alpha}}\otimes v_{\boldsymbol{\beta}}\mapsto v_{\boldsymbol{\beta}'}\otimes X_i^- v_{\boldsymbol{\alpha}'}$ in order to finally reach the target vector $v_{\boldsymbol{\beta}'}\otimes v_{\boldsymbol{\alpha}'}$ by the subsequent action of $q^{H_i}\otimes X_i^+$. The element Z_i can explicitly be written down for each choice of $\alpha_i,\beta_i,\ldots,\alpha'_{i+1},\beta'_{i+1}$ by substituting (3.3) and (7.9). There are 2^8 cases in total, and most of them are identically 0. By a direct calculation one can check that all the nontrivial cases become 0 by using the relation (3.1) and (7.1). In short, our claim is $Z_i=0$, which is independent of the representation of $\mathcal{A}$ and also of the bra and ket vectors in (5.4) and (5.5) .

• *Proof of* (6.11) *with* $g=X_n^+$ *for* $B_n^{(1)}$ *and* $D_{n+1}^{(2)}$. The relevant R matrices in (7.2) and (7.3) are $\mathcal{R}^{s,t}(x)$ with $t=1$. Thus we are to show

$$\check{\mathcal{R}}^{s,1}(x)(q^{H_n}\otimes X_n^+ + X_n^+\otimes q^{-H_n})=(q^{H_n}\otimes X_n^+ + X_n^+\otimes q^{-H_n})\check{\mathcal{R}}^{s,1}(x), \tag{7.11}$$

where X_n^+ and H_n are specified in (6.4) . By the same argument as before, we are to check that $\langle MZ_n\rangle_{s1}=0$ for any element $M\in\mathcal{A}$, where Z_n is given by

$$\begin{aligned}Z_n=&L(\alpha'_n,\beta'_n|q^H\alpha_n,X^+\beta_n)+L(\alpha'_n,\beta'_n|X^+\alpha_n,q^{-H}\beta_n)\\ &-L(X^-\alpha'_n,q^H\beta'_n|\alpha_n\beta_n)-L(q^{-H}\alpha'_n,X^-\beta'_n|\alpha_n,\beta_n)\in\mathcal{A}.\end{aligned} \tag{7.12}$$

There are 2^4 Z_n's depending on the choices of $\alpha_n,\ldots,\beta'_n$. Writing them out one finds that they all vanish by virtue of (3.1), except the two nontrivial cases proportional to $\mathbf{a}^\pm-1-iq^{\mp1}\mathbf{k}$. Thus $\langle MZ_n\rangle_{s1}=0$ follows from (4.3) and (7.1).

• *Proof of* (6.11) *with* $g=X_n^+$ *for* $D_n^{(1)}$. The relevant R matrix in (7.4) is $\mathcal{R}^{2,2}(x)$. Thus we are to show

$$\check{\mathcal{R}}^{2,2}(x)(q^{H_n}\otimes X_n^+ + X_n^+\otimes q^{-H_n})=(q^{H_n}\otimes X_n^+ + X_n^+\otimes q^{-H_n})\check{\mathcal{R}}^{2,2}(x), \tag{7.13}$$

where X^+_n and H_n are specified in (6.5) . As before we are to show $\langle MZ'_n\rangle_{22} = 0$ for any $M \in \mathcal{A}$, where

$$\begin{aligned} Z'_n =&L(\alpha'_{n-1}, \beta'_{n-1}|q^H\alpha_{n-1}, X^+\beta_{n-1})L(\alpha'_n, \beta'_n|q^H\alpha_n, X^+\beta_n)\\ &+L(\alpha'_{n-1}, \beta'_{n-1}|X^+\alpha_{n-1}, q^{-H}\beta_{n-1})L(\alpha'_n, \beta'_n|X^+\alpha_n, q^{-H}\beta_n)\\ &-L(X^-\alpha'_{n-1}, q^H\beta'_{n-1}|\alpha_{n-1}, \beta_{n-1})L(X^-\alpha'_n, q^H\beta'_n|\alpha_n, \beta_n)\\ &-L(q^{-H}\alpha'_{n-1}, X^-\beta'_{n-1}|\alpha_{n-1}, \beta_{n-1})L(q^{-H}\alpha'_n, X^-\beta'_n|\alpha_n, \beta_n) \in \mathcal{A}. \end{aligned} \tag{7.14}$$

There are 2^8 Z'_n's depending on $\alpha'_{n-1}, \dots, \beta_n$. Due to (3.1) and (7.1), they all vanish except the two cases proportional to $1 - (\mathbf{a}^\pm)^2 + q^{\mp 2}\mathbf{k}^2$. Thus $\langle MZ'_n\rangle_{22} = 0$ follows from (3.1), (7.1) and the first relation in (4.5) with $x = 1$.

The proof of (6.11) has been finished. Next we proceed to (6.12) concerning the 0-action.

• *Proof of* (6.12) *for* $B^{(1)}_n$ *and* $D^{(1)}_n$. We use (6.6) and (6.7). We are to check that $\langle Z_0N\rangle_{2t} = 0$ for $t = 1\,(B^{(1)}_n)$, $t = 2\,(D^{(1)}_n)$ and for any $N \in \mathcal{A}$, where

$$\begin{aligned} Z_0 =&L(\alpha'_1, \beta'_1|q^{-H}\alpha_1, X^-\beta_1)L(\alpha'_2, \beta'_2|q^{-H}\alpha_2, X^-\beta_2)\\ &+xL(\alpha'_1, \beta'_1|X^-\alpha_1, q^H\beta_1)L(\alpha'_2, \beta'_2|X^-\alpha_2, q^H\beta_2)\\ &-xL(X^+\alpha'_1, q^{-H}\beta'_1|\alpha_1, \beta_1)L(X^+\alpha'_2, q^{-H}\beta'_2|\alpha_2, \beta_2)\\ &-L(q^H\alpha'_1, X^+\beta'_1|\alpha_1, \beta_1)L(q^H\alpha'_2, X^+\beta'_2|\alpha_2, \beta_2) \in \mathcal{A}. \end{aligned} \tag{7.15}$$

There are 2^8 Z_0's depending on the choices of $\alpha_1, \dots, \beta'_2$. Due to (3.1) and (7.1), they all vanish except the following:

$$\mathbf{a}^+ - x\mathbf{a}^-, \quad \mathbf{a}^\pm\mathbf{k} + (xq^2)^{\pm 1}\mathbf{k}\,\mathbf{a}^\mp, \quad x^{\mp 1}(\mathbf{a}^\pm)^2 - 1 - q^{\pm 2}\mathbf{k}^2. \tag{7.16}$$

The leftmost one makes zero contribution owing to the second relation in (4.5). Therefore as far as the action on $\langle\overline{\chi}_2(x)|$ from the right is concerned, one can write $\mathbf{a}^+ \equiv x\mathbf{a}^-$. Then the remaining ones in (7.16) are $\equiv$ to $\mathbf{a}^\pm\mathbf{k} + q^{\pm 2}\mathbf{k}\,\mathbf{a}^\pm$ and $\mathbf{a}^\mp\mathbf{a}^\pm - 1 - q^{\pm 2}\mathbf{k}^2$. These combinations are indeed 0 because of (3.1) and (7.1). Note that this arguement holds either for $t = 1$ or $t = 2$ which concerns the choice of the ket vectors in (5.4) and (5.5).

• *Proof of* (6.12) *for* $D^{(2)}_{n+1}$. We use (6.8) and (6.9). We are to check that $\langle Z'_0N\rangle_{11} = 0$ for any $N \in \mathcal{A}$, where

$$\begin{aligned} Z'_0 =&L(\alpha'_1, \beta'_1|q^{-H}\alpha_1, X^-\beta_1) + xL(\alpha'_1, \beta'_1|X^-\alpha_1, q^H\beta_1)\\ &-xL(X^+\alpha'_1, q^{-H}\beta'_1|\alpha_1, \beta_1) - L(q^H\alpha'_1, X^+\beta'_1|\alpha_1, \beta_1) \in \mathcal{A}. \end{aligned} \tag{7.17}$$

There are 2^4 Z'_0's depending on the choices of $\alpha_1, \dots, \beta'_1$. Due to (3.1) and (7.1), they all vanish except $\mathbf{a}^\pm - x^{\pm 1}(1 + iq^{\pm 1}\mathbf{k})$. Thus $\langle\overline{\chi}_1(x)|Z'_0 = 0$ holds thanks to (4.4).

Our proof of (6.12) is finished and thereby Theorem 7.1 is established. □

Appendix A. Explicit formula of $\mathcal{R}$

For reader's convenience, we quote from [11] the explicit formula for $\mathcal{R} = \mathcal{R}_{1,2,3}$ in Theorem 3.1 in a form free from implicit poles. Define its matrix elements by

$$\mathcal{R}^{n'_1,n'_2,n'_3}_{n_1,n_2,n_3} = \langle n_1, n_2, n_3|\mathcal{R}|n'_1, n'_2, n'_3\rangle \quad (n_i, n'_i \in \mathbb{Z}_{\ge 0}), \tag{A.1}$$

where $\langle n_1, n_2, n_3| = \langle n_1| \otimes \langle n_2| \otimes \langle n_3|$ and similarly for $|n'_1, n'_2, n'_3\rangle$. Then we have

$$\mathcal{R}^{n'_1,n'_2,n'_3}_{n_1,n_2,n_3} = \sqrt{\frac{(q^2;q^2)_{n'_1}(q^2;q^2)_{n'_2}(q^2;q^2)_{n'_3}}{(q^2;q^2)_{n_1}(q^2;q^2)_{n_2}(q^2;q^2)_{n_3}}}\,\delta_{n_1+n_2,n'_1+n'_2}\delta_{n_2+n_3,n'_2+n'_3}\,\overline{\mathcal{R}}^{n'_1,n'_2,n'_3}_{n_1,n_2,n_3}, \tag{A.2}$$

$$\begin{aligned}\overline{\mathcal{R}}^{n'_1,n'_2,n'_3}_{n_1,n_2,n_3} &= (-1)^{n'_2}\frac{q^{n_1n_3+n'_2(n_1+n_3+1)}}{(q^2;q^2)_{n'_2}}\\ &\times\sum_{k=\max(0,n'_2-n_2)}^{\min(n_1,n_3,n'_2)}\frac{(q^{-2n_1};q^2)_k(q^{-2n_3};q^2)_k(q^{-2n'_2};q^2)_k(q^{2(n_2-n'_2+k+1)};q^2)_{n'_2-k}q^{2k}}{(q^2;q^2)_k}.\end{aligned} \tag{A.3}$$

This formula is equivalent to [11, eq. (59)] without the sign $(-1)^{n_2}$. Removing it corresponds to setting $\varepsilon = -1$ in [11, eq. (22)], which is necessary to match eq. (3.6) in this paper. The matrix elements enjoy the following symmetry:

$$\mathcal{R}^{n'_1,n'_2,n'_3}_{n_1,n_2,n_3} = \mathcal{R}^{n_1,n_2,n_3}_{n'_1,n'_2,n'_3} = \mathcal{R}^{n'_3,n'_2,n'_1}_{n_3,n_2,n_1}.$$

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References

[1] A. B. Zamolodchikov, *Tetrahedra equations and integrable systems in three-dimensional space*, Soviet Phys. JETP **52** 325-336 (1980).
[2] A. B. Zamolodchikov, *Tetrahedron equations and relativistic S matrix of straight strings in* $(2+1)$-*dimensions*, Commun. Math. Phys. **79** 489-505 (1981).
[3] R. J. Baxter, *Exactly solved models in statistical mechanics*, Dover (2007).
[4] V. V. Bazhanov and S. M. Sergeev, *Zamolodchikov's tetrahedron equation and hidden structure of quantum groups*, J. Phys. A: Math. Theor. **39** 3295-3310 (2006).
[5] R. J. Baxter, *The Yang-Baxter Equations and the Zamolodchikov Model*, Physica **18D** 321-247 (1986).
[6] V. V. Bazhanov and R. J. Baxter, *New solvable lattice models in three-dimensions*, J. Stat. Phys. **69** 453-585 (1992).
[7] M. Okado, *Quantum R matrices related to the spin representations of* B_n *and* D_n, Commun. Math. Phys. **134** 467-486 (1990).
[8] M. Jimbo, *Quantum R matrix for the generalized Toda system*, Commun. Math. Phys. **102** 537-547 (1986).
[9] V. V. Bazhanov, *Trigonometric solution of triangle equations and classical Lie algebras*, Phys. Lett. **B159** 321-324 (1985).
[10] S. M. Sergeev, *Supertetrahedra and superalgebras*, J. Math. Phys. **50** 083519 (2009).
[11] V. V. Bazhanov, V. V. Mangazeev and S. M. Sergeev, *Quantum geometry of 3-dimensional lattices*, J. Stat. Mech. P07006 (2008).
[12] J. M. Maillet and F. Nijhoff, *Integrability for multidimensional lattices*, Phys. Lett. **B224** 389 (1989).
[13] I. G. Korepanov, *Tetrahedral Zamolodchikov algebras corresponding to Baxter's L-operators*, J. Stat. Phys. **71** 85-97 (1993).
[14] I. G. Korepanov, *Algebraic integrable dynamical systems,* 2+1 *dimensional models on wholly discrete space-time, and inhomogeneous models on 2-dimensional statistical physics*, arXiv:solv-int/9506003 (1995).
[15] R. M. Kashaev, I. G. Korepanov and S. M. Sergeev, *The functional tetrahedron equation*, Teor. Mat. Fiz. **117** 370-384 (1998).
[16] S. M. Sergeev, *Tetrahedron equations, boundary states and the hidden structure of* $U_q(D^{(1)}_n)$, J. Phys. A: Math. Theor. **42** 082002 (2009).
[17] G. E. Andrews, *The theory of partitions*, Cambridge Univ. Press (1984).
[18] V. G. Drinfeld, *Quantum groups*, In Proceedings of the International Congress of Mathematicians, pp798-820, New York: Berkeley (1986).
[19] V. G. Kac, *Infinite dimensional Lie algebras*, third ed., Cambridge University Press (1990).

[20] Y. Koga, *Commutation relations of vertex operators related with the spin representation of* $U_q(D_n^{(1)})$, Osaka J. Math. **35** 447-486 (1998).

[21] N. Yu. Reshetikhin, *Algebraic Bethe ansatz for* $SO(n)$ *invariant transfer-matrices*, Zapiski nauch. LOMI **169** 122-140 (1988) (in Russian).

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